

Classical approaches to Higgs mechanism

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Abstract

The standard approach to Higgs mechanism is based on the existence of unitary gauge but, unfortunately, it does not come from a coordinate change in the configuration space of the initial model and actually defines a new dynamical system. So, it is a questionable approach to the problem but it is shown here that the final result could still make sense as a Marsden-Weinstein reduced system. (This reduction can be seen as completely analogous to the procedure of obtaining the “centrifugal” potential in the classical Kepler problem.)

It is shown that in the standard linearization approximation of the Coulomb gauged Higgs model geometrical constraint theory offers an explanation of the Higgs mechanism because solving of the Gauss law constraint leads to different physical submanifolds which are not preserved by the action of the (broken) global $U(1)$ group.

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Despite the phenomenal success of the Standard Model, Higgs mechanism remains yet experimentally unverified. The current presentation of the spontaneous symmetry breaking (SSB) is still not quite convincing. It boils down to applying change of variables (which a closer inspection reveals not to be a coordinate change), in order to rearrange the quadratic terms in the Lagrangian in a form suggesting presence of certain particles and absence of others. Elimination of dynamics along the action of the global symmetry group to be broken is done by hand and without justification. It would be interesting to reanalyse the problem from purely classical viewpoint without appealing to the quantum mystique.

To be concrete we shall concentrate on the Lagrangian analysed in [1] in its form highlighting “radial-angular” decomposition:

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu R \partial^\mu R - \frac{1}{2}e^2 R^2 A_\mu A^\mu + eR^2 A_\mu \partial^\mu \theta - \frac{1}{2}R^2 \partial_\mu \theta \partial^\mu \theta - V(R^2)$$

and the corresponding Hamiltonian (when we assume the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$) is:

$$\begin{aligned} H = & \frac{1}{2} \left[\pi_R^2 + (\nabla R)^2 + \pi_{\mathbf{A}}^2 + (\nabla \times \mathbf{A})^2 - 2(\nabla A_0)^2 + R^2(e\mathbf{A} - \nabla\theta)^2 \right] + \\ & + \frac{\pi_\theta^2}{2R^2} + eA_0\pi_\theta + V(R^2) \end{aligned} \quad (1)$$

where $\pi_R = \partial_0 R$, $\pi_{\mathbf{A}} = \mathbf{E}$, $E_k = F_{0k}$ and $\pi_\theta = R^2(\partial_0 \theta - eA_0)$ and $(-+++)$ metric signature is assumed.

Most of the treatments of Higgs model make use of the so called unitary gauge defining $eB_\mu = eA_\mu - \partial_\mu \theta$ and rewriting the Lagrangian as:

$$L' = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu R \partial^\mu R - \frac{1}{2}e^2 R^2 B_\mu B^\mu - V(R^2).$$

Like any Lagrangian, L is a function on a tangent bundle $T\mathcal{M}$ and in this case the configuration space \mathcal{M} is the space of potentials A_μ and the fields R and θ ; while L' is function on the tangent bundle of the new field B_μ and the field R . The mixing of variables on the configuration and tangent space means that L' presents a new dynamical system (possibly quite sensible one) which is not equivalent to the initial one and which cannot be obtained by a mere coordinate change in the configuration space. The standard explanation is that after a local gauge transformation the Lagrangian could be rewritten

in the new form but, in general, this is not an allowed procedure. A natural question arises whether there is a more rigorous explanation of this recipe.

The present paper claims that the final result coincides with the Marsden-Weinstein (MW) reduction [2] of the initial dynamical system (and is actually analogous to the treatment of the classical Kepler problem leading to the “centrifugal” potential). To remind, when we have a group G with a Lie algebra \mathcal{G} acting on a symplectic manifold P in a (strongly) Hamiltonian manner and defining a Lie algebra homomorphism, we have a momentum mapping $\mathcal{J} : P \rightarrow \mathcal{G}^*$ given by

$$\langle \mathcal{J}(p), a \rangle = f_a(p) \quad \forall a \in \mathcal{G}$$

where f_a is the Hamiltonian function of the fundamental vector field defined by the action of a and also satisfying

$$[f_a, f_b] = f_{[a, b]} \quad \forall a, b \in \mathcal{G}$$

then the MW quotient manifold $\mathcal{J}^{-1}(\mu)/G_\mu$ has a unique symplectic structure (provided μ is weakly regular value of \mathcal{J} and the action of the isotropy group of $\mu - G_\mu$ on $\mathcal{J}^{-1}(\mu)$ is free and proper). This is a powerful method for obtaining reduced dynamics on a symplectic space starting from symplectic dynamical system with a symmetry. (We shall skip here any technicalities like which group actions admit momentum maps, Ad^* -equivariance, clean intersections etc.) In our case the group to be broken $U(1)$ acts as

$$\theta \rightarrow \theta + \phi \tag{2}$$

and the space $\mathcal{J}^{-1}(0)$ is the subspace defined by $\pi_\theta \equiv -R^2 B_0 = 0$. The group action quotienting of this space amounts to elimination of any residual θ -dependence. Reducing the Hamiltonian (1) (and assuming that the Gauss law constraint $\Delta A_0 = e\pi_\theta$ is solved) we obtain the Hamiltonian corresponding to L' in the $B_0 = 0$ gauge, in conformity with the standard interpretation.

It is noteworthy to analyse the more general case when we reduce by a nonzero value μ of the dual algebra \mathcal{G}^* . The MW quotienting would be equivalent to fixing $\pi_\theta = \text{const} \neq 0$ and again factoring out θ from the phase space and the Hamiltonian. As a result the initial potential $V(x) = -ax + bx^2$ with $a, b > 0$ will be modified by a cx^{-2} term (with $c > 0$) and this will lead to higher values of the Higgs mass without possibility for its vanishing. (This could also add a new free parameter in possible future experimental testing of the Higgs mechanism.)

This is actually not an explanation why θ -symmetry is spontaneously broken—this is just a more rigorous procedure for factoring out (θ, π_θ) degree of freedom and thus eliminating the movement along θ which would be the dynamics typical for a massless field. It is precisely this movement along the flat bottom of the potential surface which could lead to a massless (Goldstone) field. One could still ask what prevents movement in this direction and hence causing SSB. Being aware that SSB could only exist in systems with infinite degrees of freedom, one may also wonder where this property is encoded in the above mentioned procedures.

A rigorous approach to these problems could be found e.g. in [3] where a structure of Hilbert space sectors (HSS) is found in solutions of nonlinear classical relativistic field equations. Each sector is invariant under time evolution, has a linear structure and is isomorphic to a Hilbert space; and may be labeled by conserved dynamical charges. Different HSS define “dis-joint physical worlds” which could be considered as a set of configurations which are accessible in a given laboratory starting from a given ground state configuration. Then any group which maps a HSS into another HSS is spontaneously broken and only “stability” groups which map a HSS into itself would be proper symmetry groups.

Despite the nontriviality of existence of stable linear structures in the set of solutions of certain nonlinear equations and the possibility to explain in principle the existence of SSB this approach does not seem very practical. Another possible route is offered by the use of geometrical constraint theory. Higgs model is a beautiful example of a constrained system. Lagrangian does not depend on θ but only on $\partial_\mu \theta$, thus allowing solutions with θ -rotations but (θ, π_θ) degree of freedom remains coupled with the potential A_μ . The assumption that θ -rotations are frozen (and consequent writing them off) obviously seems ungrounded.

In what follows we shall return to our model in the Coulomb gauge. This case was successfully tackled [4] by linearisation of the equations leading to massive wave equation for θ . Here we shall be interested not so much in the (linearised) equations but in the symmetry breaking. We have primary constraint $\pi_0 \equiv \frac{\partial L}{\partial(\partial_0 A_0)} = 0$ and the condition of its preserving gives the Gauss law constraint equation $\Delta A_0 = e\pi_\theta$. To be precise, this equation is not the proper constraint – the submanifolds determined by its solutions will give the surfaces on which the dynamics will be constrained after factoring out the (A_0, π_0) degree of freedom. Obviously, the solutions of the equation

are

$$A_0 = G * e\pi_\theta + f$$

where G is a Green function for the Laplacian, $*$ denotes convolution i.e. $(g * h)(x) = \int g(x-y) h(y) dy$ and f is any function satisfying $\Delta f(\mathbf{x}, t) = 0$. Solutions of this equation would define different physical submanifolds labeled by solutions of this equation. After differentiation we obtain

$$\dot{A}_0 = eG * \dot{\pi}_\theta + \dot{f} = -eG * \partial_k(R^2 B_k) + \dot{f}$$

and taking into account that $\partial_k(R^2 B_k) = \Delta(R^2 \theta)$ in the linearisation approximation [4], we have:

$$\dot{A}_0 = eR^2 \dot{\theta} + \dot{f}$$

This shows that we will have different dynamics on different physical submanifolds because the general form of the “massive wave equation” for θ would be $\square \theta = e^2 R^2 \theta + e \dot{f}$ (as long as linearisation approximation could be trustworthy). More interestingly, transformations (2) does not act by lifting from the “configuration space”:

$$A_0 \rightarrow A_0 \quad \dot{A}_0 \rightarrow \dot{A}_0 + eR^2 \phi$$

and does not preserve the chosen submanifold. Transformation actions of this kind are not very typical in physics (the standard Nöther theorem, for example, assumes only lifted transformations). Thus we have a geometrical analog of HSS and its origin could be traced to the requirements to the physical constrained submanifolds in infinite dimensions [5].(In this reference one could also see how this phenomenon appears in an exactly solvable model.)

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